

Para- f -Lie Algebras

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Abstract

In this paper the notion of a para- f -Lie algebra is introduced. It is shown that a Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup if and only if its Lie algebra \mathfrak{g} is a para- f -Lie algebra.

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1 Introduction

The theory of f -structures and para- f -structures on differentiable manifolds has been developing for many years. The concept of f -structures on manifolds was presented and studied by many authors, e.g. [4], [5], and was defined as an endomorphism field f of the tangent bundle TM of a manifold M satisfying $f^3 + f = 0$.

The notion of a para- f -structure on a differentiable manifold M was introduced in [1]. Its basic properties will be presented in Section 2. Para- f -structures on Lie groups were studied in [2], where the notion of a para- f -Lie group was introduced and it was shown that the Lie group $GL(n)$ is a para- f -Lie group and can be expressed as the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup. The basic properties of para- f -Lie groups will be presented in Section 3.

Sometimes, properties of Lie groups can be studied by dealing with algebraic properties of their Lie algebras. In Section 4, the notion of a para- f -Lie algebra is defined. It turns out that to study properties of para- f -Lie groups is sufficient to study algebraic properties of their Lie algebras. By showing that

the existence of a para- f -Lie structure on a Lie group G is equivalent with the existence of a para- f -structure on its Lie algebra \mathfrak{g} , it is shown that some properties of para- f -Lie groups proven in [2] become simple consequences of that equivalence.

2 Para- f -Structures

The notion of a para- f -structure on a differentiable manifold was introduced and studied in [1]. In this section some basic definitions and properties are recalled.

Definition 2.1 *Let M be an n -dimensional differentiable manifold. If φ is an endomorphism field of constant rank k on M satisfying*

$$\varphi^3 - \varphi = 0, \quad (1.1)$$

*then φ is called a **para- f -structure** on M and M is a **para- f -manifold**.*

Definition 2.2 *A para- f -structure φ on M is **integrable** if there exists a coordinate system in which φ has constant components*

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.2)$$

where I_s is the unit $s \times s$ -matrix and $p + q = k$.

Proposition 2.1 *A para- f -structure φ on M is integrable if and only if its Nijenhuis tensor field N_φ vanishes, that is*

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] = 0 \quad (1.3)$$

where X, Y are vector fields on M .

Let $\ker \varphi = \bigcup_{m \in M} (\ker \varphi)_m$ and $\operatorname{im} \varphi = \bigcup_{m \in M} (\operatorname{im} \varphi)_m$ be the kernel and image of φ on M , respectively, where

$$(\ker \varphi)_m = \{X \in T_m M; \varphi_m(X) = 0\} \quad (1.4)$$

$$(\operatorname{im} \varphi)_m = \{Y \in T_m M; Y = \varphi_m(X) \text{ for some } X \in T_m M\} \quad (1.5)$$

are the kernel and image of φ at any point $m \in M$, respectively.

Proposition 2.2 *If $(\ker \varphi)_m = \{0\}$ for a para- f -structure φ for all $m \in M$, then φ is an almost product structure on M , that is $\varphi^2 = \operatorname{Id}$.*

Proposition 2.3 *If $(\text{im } \varphi)_m = \{0\}$ for a para- f -structure φ for all $m \in M$, then φ is the trivial para- f -structure on M , that is $\varphi = 0$.*

Proposition 2.4 *If φ is a para- f -structure on M , then*

$$\ker \varphi \cap \text{im } \varphi = \{0\} \quad (1.6)$$

Definition 2.3 *Let φ_i be a para- f -structure on a para- f -manifold M_i with $i = 1, 2$. A diffeomorphism $h : M_1 \rightarrow M_2$ is called a **para- f -map** if*

$$\varphi_2 \circ h_* = h_* \circ \varphi_1, \quad (1.7)$$

where h_* is the differential of h .

3 Para- f -Lie Groups

The notion of a para- f -Lie group was introduced in [2]. In this section all definitions and properties of a para- f -Lie group are recalled.

Definition 3.1 *Let G be a Lie group with a para- f -structure φ . If both the left multiplication L_g and the right multiplication R_g for each $g \in G$ are para- f -maps, then φ is said to be **bi-invariant**.*

Definition 3.2 *If G is a Lie group with an integrable bi-invariant para- f -structure φ , then G is called a **para- f -Lie group**.*

Proposition 3.1 *A Lie group G with a bi-invariant para- f -structure φ is a para- f -Lie group.*

Theorem 3.1 *Every para- f -Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup.*

Proposition 3.2 *The Lie group $GL(n)$ of real $n \times n$ -matrices is the para- f -Lie group.*

4 Para- f -Lie Algebras

In this section, the notion of a para- f -Lie algebra is defined. It turns out that the existence of a para- f -structure on a Lie group is reduced to a purely algebraic problem in its Lie algebra. It is shown that a para- f -structure exists on a Lie group if and only if there exists a para- f -Lie structure on its Lie algebra.

Definition 4.1 Let \mathfrak{g} be a Lie algebra and let $\varphi_0 : \mathfrak{g} \rightarrow \mathfrak{g}$ be its endomorphism satisfying

$$\varphi_0^3 - \varphi_0 = 0, \quad (3.1)$$

$$[\varphi_0(X), \varphi_0(Y)] - \varphi_0([\varphi_0(X), Y]) - \varphi_0([X, \varphi_0(Y)]) + \varphi_0^2([X, Y]) = 0, \quad (3.2)$$

then φ_0 is called a **para- f -structure** on \mathfrak{g} and \mathfrak{g} is a **para- f -Lie algebra**.

Theorem 4.1 A connected Lie group G is a para- f -Lie group if and only if its Lie algebra \mathfrak{g} is a para- f -Lie algebra.

Proof. Suppose that G is a para- f -Lie group, that is, there is an integrable para- f -structure φ on G . For $X \in \mathfrak{g}$ and $g \in G$ we have

$$L_{g*}(\varphi(X)) = \varphi(L_{g*}(X)) = \varphi(X). \quad (3.3)$$

Hence, for any $X \in \mathfrak{g}$ we have $\varphi(X) \in \mathfrak{g}$, and the restriction φ_0 of φ to \mathfrak{g} is an endomorphism of \mathfrak{g} . The endomorphism φ_0 satisfies (3.1) and (3.2) which shows that \mathfrak{g} is a para- f -Lie algebra.

Conversely, suppose that \mathfrak{g} has a para- f -structure φ_0 satisfying (3.1) and (3.2). Let $\{E_1, E_2, \dots, E_n\}$ be a basis of \mathfrak{g} . Then for any vector field X on G we can find n functions $\alpha_1, \alpha_2, \dots, \alpha_n$ on G such that X can be written uniquely as $X = \alpha^i E_i$. Now define the endomorphism φ of G as

$$\varphi : G \rightarrow G, \quad X \mapsto \varphi(X) = \alpha^i \varphi_0(E_i). \quad (3.4)$$

Then, clearly φ satisfies (1.1). On the other hand, by (3.2) we have

$$[\varphi_0(E_i), \varphi_0(E_j)] - \varphi_0([\varphi_0 E_i, E_j]) - \varphi_0([E_i, \varphi_0(E_j)]) + \varphi_0^2([E_i, E_j]) = 0, \quad (3.5)$$

and this implies (1.3), which proves that φ is integrable and G is a para- f -Lie group.

As a consequence of **Theorems 3.1** and **4.1** we obtain

Theorem 4.2 A Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup if and only if its Lie algebra \mathfrak{g} is a para- f -Lie algebra.

Example 4.1 Let $\mathfrak{gl}(n, \mathbb{R})$ be the Lie algebra of all real $n \times n$ matrices. Let

$$\varphi : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(n, \mathbb{R}), \quad X \mapsto \varphi(X) = X - \frac{1}{n} \text{trace}(X) I_n, \quad (3.6)$$

where I_n is the unit $n \times n$ -matrix. Then φ is a para- f -structure on $\mathfrak{gl}(n, \mathbb{R})$, and $\mathfrak{gl}(n, \mathbb{R})$ is the para- f -Lie algebra.

Hence, from **Theorem 4.1** $GL(n)$ is the para- f -Lie group, and this is the simplest proof of **Proposition 3.2**. Also, we have

Proposition 4.1 *The group $GL(n)$ can be expressed as the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup.*

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