Half Iterates of Formal Power Series

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Here, then, is the desired formula:

$a_0 = A_0, a_1 = \pm\sqrt{A_1}$

$$a_{n\geqslant 2} = \frac{A_n - \sum_{m=2}^{n-1} a_m \sum_{\forall i(s_i \geqslant 0) \wedge \sum_{i=1}^{n} s_i = m \wedge \sum_{i=1}^{n} i s_i = n} (s_i)! \prod_{i=1}^{n} a_i^{s_i}}{a_1^n + a_1}$$

$\{a_n\}$ represents the half iterate of $\{A_n\}$, where the sequences are interpreted as the coefficients of taylor series about a fixed point A_0.

To get the half-iterate of the exponential function, we must expand the exponential function about a fixed point. There are infinitely many such points. Here, letting λ represent the fixed point with least magnitude & positive imaginary part, we find:

$$\exp^{1/2}(x) = \lambda + \sqrt{\lambda}(x-\lambda) + \frac{\lambda}{2(\sqrt{\lambda}+\lambda)}(x-\lambda)^2 + \frac{\lambda^{3/2} - \lambda + \sqrt{\lambda}}{6(\sqrt{\lambda}+1)^2(\lambda+1)}(x-\lambda)^3 + \ldots$$

$(\lambda \approx 0.318131505204764135312654251587 + 1.337235701430689408901162143193i)$

The above series unfortunately appears to converge very slowly.

For numerical computations, it is possible to generate a few partial sums, then use the Euler transform to accelerate the convergence. Unfortunately, however, the above definition for a_n has factorial time complexity. It is not sufficiently fast for practical use: it took 825.07 CPU seconds for Mathematica 5.2 to approximate a_{12} to 11 decimal places on a Pentium 4 2.4GHz system with 512MB RAM (while it took only 0.69 second to generate a_0 through a_7). Using terms though the twelfth & taking the Euler transform of the partial sums resulted in a value for $\exp^{1/2}(0)$ accurate to only 2 decimal places.