INTERMEDIATE ALGEBRA

Notes on Chapter 7 (Sections 1 through 8)

These notes do not in any way replace your own work and study (and these are NOT the notes that you can turn in for extra credit—see the guidebook for information on how to correctly make those notes). These are important points and hints that can help you avoid some of the common mistakes that students tend to make in these sections.

SECTION 1: Polynomial Functions

1. Know the vocabulary: degree, leading coefficient, polynomial function, end behavior.

2. From example 2, make sure you can evaluate functions. This one's easy—just substitute and solve.

3. From example 3, ***WARNING:*** remember and correctly use the exponent rules.
   a. In example 3a, a^2 is cubed, resulting in a^6. It's also squared, resulting in a^4.
   b. In example 3b, (a+1) is squared. Here's where that mistake some students have been making this year is going to hurt them again, if they don't pay more attention right now. $(a+1)^2$ is **NOT** $a^2 + 1^2$. It **IS** $(a+1)(a+1)$, which multiplies out to be $a^2 + 2a + 1$.

4. From example 4, you have to be able to state the number of zeros when you see a graph. It's lots easier than you think—the zeros are when y = 0, which means the graph is touching or crossing (intersecting) the x-axis.

5. End behavior: Think this way: Where does the y go?
   a. What happens to the **y** as the **x** goes way out there to negative infinity ($x \to -\infty$)?
   b. What happens to the **y** as the **x** goes way out there to positive infinity ($x \to \infty$)?

6. Even and odd degree:

   a. All **EVEN** degree polynomials are basically shaped like a parabola, so both ends ALWAYS **both** go up or **both** go down.
   b. All **ODD** degree polynomials have the basic S-shape, so both ends ALWAYS go opposite directions (one will be going up to positive infinity, and one will be going down to negative infinity).

<u>SECTION 2</u>: Graphing Polynomial Functions

1. Know the vocabulary: Location Principle, relative maximum, relative minimum.

2. Locate zeros: In example 2, you see that the function intersects the x-axis, but not at an integer. All three zeros are going to be decimals or fractions, not nice and easy integers. Example 2 shows how to figure out which integers the zeros are between. It requires you do the following
   a. Make an x-y table. Put in enough x values so that you can sketch a simple, basic graph of the function. You should then end up with the three places where the y-values change their signs (or just two or one place; the graph can show you if you only touch in one place, like in example 3).
   b. Look at the y-values to identify where the sign changes. Notice in example 2 it's at –7 and 2, and at 1 and –4, and also at –2 and 17.
   c. Go to the corresponding x-values. THOSE are the values between which you have zeros. ***<u>WARNING:</u>*** Many students can identify the y-values that result in sign changes, but then they forget that it's the x-values that are the actual answers.

3. Relative maximums and relative minimums:
   a. Know what they look like on a graph.
   b. Don't get maximum and minimum mixed up! Maximums are as high as one curve goes up, and minimums are as low as one curve goes down. They make sense, if you think about it: the maximum value on that curve before it goes back down, and the minimum value on another curve before it goes back up. Yet these are relative, because the end behavior of the graph results in true maximums that are much higher, and true minimums that are much lower.

<u>SECTION 3</u>: Solving Equations Using Quadratic Techniques

1. Know the vocabulary: quadratic form.

2. Example 1 shows that you have to manipulate the exponents so that you can get the quadratic form.
   a. You MUST end up with the x^2 in the first term, x^1 in the second term (we don't show the 1 in the exponent, even though it's really there), and no x (just the constant) in the third term.
   b. This means you have to figure out how to work the exponents so that you can get that square for the first term. It's like a puzzle or game, and you have enough math knowledge to be able to figure out this puzzle.
   c. Most of the time, the exponent of the middle term is your key. That exponent squared should give you the correct form for the first term's variable. Look at examples 1a and 1d; that's how they were figured out.
   d. Example 1b takes more thinking, because there isn't a second term (that doesn't mean it can't be a quadratic, that just means that the coefficient of the second term is 0, so it goes away). So, you have to look at the first term and figure out, using the rules of exponents, how to make that exponent into something times 2. In example 1b, 6 =3*2, so $x^6 = (x^3)^2$.

3. You're going to have to solve a polynomial equation on the test, similar to example 2. So, look closely at what they do.
   a. The end result is to get the zeros (factor the polynomial until you can get x =…, as you see in examples 2a and 2b).
   b. Example 2a shows you that you have to:
      i. write the polynomial expression in quadratic form.
      ii. factor it.
      iii. factor it some more. (Sounds like reviewing factoring might be a good idea.)
      iv. State that the solutions are …(in this case, –3, –2, 2, and 3).
   c. Example 2b shows that you need to remember things like:
      i. **factoring** the sum of two cubes (first recognize that it's the sum of two cubes, or if it's the difference of two squares, or whatever it might be).
      ii. **zero product property** (that's the one that lets you split things up to equal zero, because if ab=0, either a or b or both MUST be zero. Note that this ONLY works when two (or more) things equal zero, nothing else).
      iii. **the quadratic formula**. As some students found out on the last test, memorize means exactly that—**get every detail exactly right**: exponents, negatives, plus/minus, multiplying correctly, dividing/reducing correctly, everything.

<u>SECTION 4</u>: : The Remainder and Factor Theorems

1. Know the vocabulary: Remainder Theorem, synthetic substitution, depressed polynomial, Factor Theorem.

2. The Remainder Theorem harkens back to your days doing long division and ending up with a remainder (such as 7 divided by 3 equals 2 remainder 1) So, you can divide a polynomial by a number to see if it's a really a factor (such as 6, which is x – 6).
    a. Example 1 method 1 shows that you can do this with synthetic division so that you don't have to calculate 64 by hand.
    b. Since I let you use calculators, direct substitution (example 1 method 2) is worth your consideration.
    c. The point of the Remainder Theorem is to get the value of the remainder.

3. Example 2 shows that you can determine if a number really is a factor. But here you **must** use synthetic division because you have to get the depressed polynomial and then factor that, too.
    a. Factor Theorem summary: (x – a) is a factor of a polynomial if the a *(notice that it's the opposite sign, because x – a = 0 means that x = +a)* has a remainder of zero when you substitute it into the polynomial (or if you do synthetic division, if necessary).
    b. Important note: If the depressed polynomial is a quadratic (meaning that the degree is 2), then you can factor the easy way—quadratic formula, or the "multiply to c and add to b" method…whatever works to correctly factor it.
    c. You are going to have a test problem like example 2, so make sure you practice problems similar to this, a lot and until you really can do this easily and well.

SECTION 5: Roots and Zeros

(This section has three pages because there's more to learn in this section than in any of the others, so carefully study all parts of this.)

1. Know the vocabulary: Fundamental Theorem of Algebra, Complex Conjugates Theorem. Also: zeros, factors, roots, solutions, conjugate.

2. Zeros/roots/solutions are essentially the same things (where the graph of a function intersects the x-axis, meaning that y = 0, and these are the x-values that cause y to be zero). If the zero/root/solution is an imaginary number (with *i*), then it's still considered a zero/root/solution, but that particular zero/root/solution does not actually intersect the x-axis.

3. A factor of a polynomial is a zero (let's say a zero of one function is 3) that is in the form x + 3. Notice that it's the opposite sign, because x – 3 = 0 means that x = 0.

4. The Fundamental Theorem of Algebra means you can be sure that every polynomial has a zero/root/solution, as long as the degree is greater than zero (remember that $x^0 = 1$, so you need at least a degree of one to actually have an x, and thus be able to actually solve for x. One main implication of the Fundamental Theorem of Algebra is that the degree of a polynomial tells you how many zeros/roots/solutions you should have for that polynomial. So a polynomial that has a degree of 7 (because x^7 is the highest variable-exponent combination) has 7 roots. Some of those roots might be imaginary, and some might even be double roots, but you can be sure there are 7.

5. From example 1, you will have one test question that will make you find the roots. Look at the differences, so you can recognize the types of things you might see and might have to do:
    a. Example 1a is simple. You've been doing these for quite a while.
    b. Example 1b is where you have double roots.
        i. $(x - 4)^2$ means (x – 4)(x – 4) = 0, so you get x = 4 and x = 4, but since they're the same, they're double roots (meaning the graph touches the x-axis at that point…see page 295, in the Key Concept box at the top. When it shows one real solution, that means that the vertex of the parabola just touches the x=axis at that point (it's almost like the parabola's two zeros kept moving along the x-axis until they merged together into one).
        ii. Note that first you have to correctly factor the polynomial; in this case, it's a trinomial square, where the first and last terms are perfect squares.
    c. Example 1c shows that if you might end up with a square root as the solution.
        i. You first have to correctly factor the polynomial. In this case, it's simply recognizing and factoring out the greatest common factor, which thus becomes one of the factors in its own right.
        ii. Example 1c reminds you that you have to put plus and minus in front when you take the square root of both sides on the equation, since x^2 could have two positives making it up, or two negatives, since two negatives multiply to make a positive.
        iii. Example 1c also shows that you might end up with imaginary roots (complex numbers). Again, remember your prior learning (this time from section 5-9).

d. Example 1d also shows you that you might end up with imaginary roots.
   i. Again, you first have to correctly factor the polynomial. In this case, you have to recognize that the polynomial is a difference of two squares ($A^2 – B^2$), so it automatically can be factored into (A + B)(A – B). And then you can do it again to the $x^2 – 1$.
      1. ***WARNING:*** Remember that this automatic method is only for the difference of two squares, not the sum of two squares.
      2. ***WARNING:*** If you've already forgotten the factoring techniques from month 3 (section 5-4), go back and memorize them again. This is one of those things that comes back again and again, and you really are supposed to remember the things you learned earlier this year—especially in math!

6. Descartes' Rule of Signs

Here's the simple way of understanding this and how to do it: Descartes' Rule of Signs allows you to figure out how what **types** of zeros a polynomial has: positive zeros, negative zeros, and imaginary zeros. All you have to do is look at the number of sign changes, then do f(–x) and look at those number of sign changes, go down by two each time, and see how many imaginary zeros you are then forced to have. Example 2 on page 373 is actually an excellent example. Let's go through it (it's essentially three steps):

a. Step 1: **count the sign changes** in the given polynomial. There are 4 sign changes, and Descartes' Rule of Signs says that means you have either that 4 positive zeros, or down by two, and two, and two. So you have either 4 or 2 or 0 positive zeros.

b. Step 2: **change the polynomial and recount the sign changes**. You make all the x's into –x's. That means that if there's an x^3, you write it as $(–x)^3$. This results in (–x)(–x)(–x), and that results in $–x^3$. That doesn't happen with the even exponents, because two negatives multiply to make a positive, just like four negatives or six negatives, and so on. Now, since the odd exponent variables are going to have their signs switched, you recount the sign changes. In this example, you have 1 sign change for f(–x), which means you must have 1 negative zero for this polynomial (you can't go down by 2 this time, because that would put you at less than zero zeros).

c. Step 3: **set up a chart** like is shown at the end of example 2. The number of imaginary zeros is forced, because there must be five zeros every single time, so after you've added up the possible positive zeros and the possible negative zeros, you're stuck with either 0, 2, or 4 imaginary zeros.

d. If a question asks you how many imaginary zeros a polynomial might have, you either have to go all the way through solving it (which might not be practical if you're not given enough information such as one of the factors), or you have to do this Descartes' Rule of Signs procedure, and at the end you'll be able to determine from your chart just how many imaginary zeros are possible. If you're only asked how many positive or how many negative zeros there might be, you won't have to do the chart, but make sure you remember the part of Descartes' Rule of Signs that says that the possible positive or negative zeros can be reduced by two, and two again, until you get to zero. ***(On the test, you have to go all the way to the chart, because you're going to have to find the possible number of imaginary zeros also.)***

7. **Actually find the zeros.** Example 3 shows that you can (and in fact, HAVE TO) use synthetic division to find the zeros of a polynomial.
    a. Start with simple numbers as possible zeros (the next section shows you how to determine the set of possible zeros, without just guessing like you have to do here).
    b. Then set up a synthetic division chart. Notice what they do with the synthetic division chart on for example 3. They don't show the in-between numbers that you add to the coefficients; they only show the sums which become the depressed polynomial.
    c. They do show the remainder, and the whole point is to get a remainder of 0. That means the number you guessed really is a zero.
    d. Now you write out the depressed polynomial and factor that.
        i. If the polynomial has a degree of 3 (so you have an x^3 as the highest variable-exponent combination) and there are 4 terms, you can use the regrouping technique.
        ii. If the polynomial has a degree of 2, you can always fall back on the quadratic formula for factoring, unless you can see the polynomial can be more easily factored (such as with $A^2 – B^2$ or other factoring techniques).

8. **Write a polynomial function.**
    a. You may remember doing this in Algebra 1. Given the solutions 2 and 5, write a quadratic function. You do this by placing the solutions in factored form, and then multiplying them (using FOIL for a quadratic): (x –2)(x – 5); $f(x)$ = x2 – 7x + 10.
    b. When one of the zeros is an imaginary number, the negative of that imaginary number **AUTOMATICALLY** is also a zero. Look at example 4. You're given 3 and (2 – i) as two of the zeros. Automatically, you know that the conjugate, (2 + i) is also a zero.
    c. Now you just put them into factor form and then multiply them out.
    d. ***WARNING***:
        i. When putting zeros into factored form, remember that it's x – that number. So the 3 becomes (x – 3), the (2 – i) becomes (x – (2 – i)) and the (2 + i) becomes (x – (2 + i)).
        ii. Here are the steps to making the polynomial (The textbook can be confusing):
            1. (x – 3)(x – (2 – i))(x – (2 + i))
            2. (x – 3)(x – 2 + i))(x – 2 – i)) *In step 2, see how the signs changed when we remove the parentheses inside the second and third factors.*
            3. (x – 3)((x – 2) + i))((x – 2) – i)) *In step 3, we group the first two terms inside the second and third factors.*
            4. $(x – 3)(x – 2)^2 – i^2)$ *After grouping in step 3, we can see the (A – B)(A + B) combination for the second and third factors, so we can automatically make those two factors into ($A^2 – B^2$). Note that (x – 2) is a term by itself, so it counts as "A."*
            5. $(x – 3)(x^2 – 4x + 4 – (–1))$ *Now we just do algebraic actions; square the (x–2) and make the i^2 into –1, since that's what it is, by definition.*
            6. $(x – 3)(x^2 – 4x + 5)$ *Simplify. Then multiply:* $x^2 – 7x^2 + 17x – 15$.
    e. ***WARNING***: Don't be thrown off when you see something like 8i as a factor; the conjugate is –8i, and so you proceed as above. The (2 – i) factor is more complicated, so it made a good example—but the concept is the same.

<u>SECTION 6</u>: Rational Zero Theorem

1. The Rational Zero Theorem allows you to make a list of all the possible zeros of a function. This is helpful because it gives you some numbers to choose from…then you have to try them out to find one that really is a zero—and that one will lead you into a depressed polynomial which you can factor immediately or else repeat the process to get more factors.

2. The Key Concept in the middle of page 378 can seem confusing, so here's a condensed, less confusing version: The Rational Zero Theorem says that the zeros of a function can be found by finding all the factors of the **constant at the end** of a polynomial and dividing them by all the factors of the **coefficient of the first term**. Example:
   a. $qx^5 + 93x^4 – 47x^3 + 385x^2 + 10x – p$. All that matters for the Rational Zero Theorem is the constant (the p) and the coefficient of the first term (the q).
   b. Find all the factors of p.
   c. Then find all the factors of q.
   d. Then take each factor of p, and divide by the first factor of q, and then by the second factor of q, and then by the third factor of q, and then by the fourth factor of q, and then...
   e. Ummm, I think you've got the idea. Thankfully, the q doesn't often have lots of factors, and many factors reduce to the same thing, which makes it a little easier.
   f. Let's look at example 1 on page 378. All the factors of 9 (plus or minus, for each) get divided by the factors of the 2. You end up with 12 possible zeros. If there are any rational zeros, they must come from that set of 12 possible zeros.

3. Examples 2 and 3 show you how to figure out which one of the possible zeros is really one of the rational zeros of the polynomial. Let's look more closely at example 3.
   a. The book tells you that there are 32 possible zeros (it already did the Rational Zero Theorem procedure for you—how nice of it).
   b. You have to simply pick one and see if it works, and if not, try another—and another, and another, and so on—until you get one that works (you only need one, most times).
   c. You apply what you learned from the Factor Theorem in 7-4, example 2 (page 367), where you do synthetic division and if the remainder is zero, the number really is a factor. Plus, by doing synthetic division you get a depressed polynomial which you can further factor and get all the rest of the factors without have to do the p/q thing again.
   d. Example 3, page 379, uses a chart to keep track of the synthetic dividing with each possible factor until it gets a true factor where the remainder is zero. The green line on top shows the coefficients of the polynomial. The yellowish lines show the results of the synthetic division. Notice that the chart does not show the multiplying parts of synthetic division. Since I can't always keep track of those in my mind when I'm doing synthetic division, I will actually write them on the chart, very small, when I'm doing my own chart for a problem like this. I recommend that you do this also, so you don't make mistakes keeping track in your mind.
   e. Once you find a true rational zero, make the depressed polynomial. If its degree is 2, you can factor it normally (quadratic formula always works) and get the other factors. If the degree is 3, you might be able to do the grouping method and get the other factors. If not, you might have to repeat the whole procedure to get a factor and another depressed polynomial.

<u>SECTION 7</u>: Operations on Functions

1. This is actually the easiest section in the chapter. There are five things you have to learn to do, and four of them are the basic operations of add, subtract, multiply, divide. Given two polynomials, just do the operations to them. The only things to watch out for:

    a. When subtracting, make sure that you get your negatives right. In example 1b, page 383, you're subtracting 4x + 5, so the 4x becomes –4x and the 5 becomes –5. This is because you have – (4x + 5), which is really –1(4x + 5), and you distribute the – 1 to both terms inside the parentheses: –1 • 4x and –1 • 5, making –4x – 5. **GET THIS RIGHT, EVERY TIME.**

    b. When dividing, be careful which is numerator and which is denominator. You can't switch them like you can when adding or multiplying.

2. Composition of Functions is a bit harder, but not much. All you really have to do is put the second function into the first. Look at example 4a, page 385.

    a. *f* o *g* means to take *g* and put it inside *f*. That really means to take the whole function of *g* and put it in place of every x that makes up *f*. $g(x) = x^2 + x - 1$, and that goes into the place of the x. Since $f(x) = x + 3$, $f \circ g = (x^2 + x - 1) + 3$. Then you simplify it, which in this case is simply combining the like terms of –1 and 3, so the final answer is $f \circ g = x^2 + x + 2$.

    b. Note that sometimes you might have to multiply, as in example 4b. Here, the x + 3 gets put into the x's of *g*(*x*). You have to square the x + 3. (***<u>WARNING</u>***: remember — again—that squaring x + 3 does NOT equal x2 + 9. You have to multiply out (x+3)(x+3) and use FOIL, and get the middle term, too.)

    c. ***<u>WARNING</u>***: pay attention to the order you're asked for. *f* o *g* and *g* o *f* are not the same thing, and you must do them in the correct order.

    d. ***<u>WARNING</u>***: (This is the biggest warning in this section, because close to half the students make this mistake every year.) You must distinguish between the solid dot for multiplication (•) and the little circle for composition (o). Don't do the wrong operation; the symbols are your clues for what to do, and you need to remember and do them correctly.

<u>SECTION 8</u>: Inverse Functions and Relations

1. Know the vocabulary: inverse relation, inverse functions, one-to-one.

2. The basic thing to know about inverse is that you switch the x and y values for each ordered pair. The graphical version of this is reflecting the graph of the relation over the line y= x, as shown in example 1 (page 390 in the textbook).

3. Memorize that f^{-1} means the inverse of f.

4. Example 2 shows you how to find an inverse: just switch the x and y, and then solve for y so that it's back into the form of y =, which is always the way you want to see a function written.

5. Verify that two functions are inverses: See the Key Concept on the bottom of page 391 and the corresponding example, which is example 3 on the top of p. 392.
    a. You have to do two compositions (as you learned in 7-7), and each of them should come out to be x. If either doesn't come out to be x, the two functions are not inverses of each other.
    b. ***<u>WARNING</u>***: Be very careful when doing the calculations. Most students get the point of doing this, but then they make little errors and think two functions are inverses when they're not, or think two functions are not inverses when they are.